<u>Differential Equations involving more difficult manipulation</u>

A) Formulating the equation from a statement

Example: Winston is a small town in Southland which has been diminishing in size for some years. The Town Council has the data from the last census figures as shown to 3 sf in the table below:

Year	1991	1996	2001	2006
Population	2470	2350	2230	2120

The council believes that the rate of population decline is proportional to the size of the population itself, and wants to use this theory as a way of predicting the town's population in 2015 for purposes of council decision-making.

Using the data as a way of forming an appropriate equation for the council's theory, predict the Winston population in 2015.

From the council's theory: $\frac{dP}{dt} = kP \quad \rightarrow \quad P = Ae^{kt}$ (general solution)

- To find the constant A, substitute in one set of points:
  When $t = 0, P = 2470$:
  $\rightarrow 2470 = Ae^{k(0)} = A \cdot 1 = A$
  $\rightarrow A = 2470$
  The equation is now $P = 2470e^{kt}$
- To find the constant k, substitute in another set of points:
  When $t = 5, P = 2350$: (this is for year 1996, with reference to 1991 being $t = 0$)
  $\rightarrow 2350 = 2470e^{k(5)}$
  $\rightarrow e^{5k} = 0.9514 \ldots$
  $\rightarrow k = -0.010$ (3dp)
  The equation is now $P = 2470e^{-0.010t}$

The equation should be tested on the other 2 set of the figures to check that it holds. $P(10) = 2230$ (3sf) and $P(15) = 2130$ (3sf) which is a little out but close enough to accept the equation for the intended purposes:

The prediction for the population of Winton in 2015 ($t = 24$) would be:
$P = 2470e^{-0.010t}$
$P = 2470e^{-0.010(24)}$
$P = 1943$
$P = 1940$ (3sf)

B) When separation of variables is not straightforward

Example: Solve the equation $\frac{dy}{dx} - 4xy = x$

$\frac{dy}{dx} = 4xy + x$ (make $\frac{dy}{dx}$ the subject)

$\frac{dy}{dx} = x(4y + 1)$ (take out common factor x)

$\frac{1}{4y+1}\ dy = x\ \ dx$ (separate the variables)

$\int \frac{1}{4y+1}\ dy = \int x\ \ dx$ (integrate)

$\frac{1}{4}\ln|4y + 1| = \frac{x^2}{2} + c$

$\ln|4y + 1| = 2x^2 + d$ (replace old constant with new constant as value changes)

$4y + 1 = e^{2x^2+d}$

$4y + 1 = e^{2x^2} \cdot e^d$

$4y + 1 = Ae^{2x^2}$ (replace the constant e^d with the constant A)

$y = Be^{2x^2} - \frac{1}{4}$